

PERIODICITY OF ALMOST PERIODIC ZERO SETS AND ALMOST PERIODIC FUNCTIONS

S.YU. FAVOROV

Abstract. Whenever all differences between zeros of two holomorphic almost periodic functions in a strip form a discrete set, then both functions are infinite products of periodic functions with commensurable periods. In particular, the result is valid for some classes of Dirichlet series.

AMS Mathematics Subject Classification: 30B50, 42A75

Keywords: quasipolynomial, periodic function, zero set, Dirichlet series, almost periodic holomorphic function.

It was proved in [1] that each quasipolynomial

$$(1) \quad Q(z) = \sum_{n=1}^N a_n e^{\lambda_n z}, \quad \lambda_n \in \mathbb{R}, \quad a_n \in \mathbb{C}.$$

with a discrete set of differences between its zeros is periodic up to a multiplier without zeros, i.e., is of the form

$$(2) \quad Q(z) = C e^{\beta z} \prod_{k=1}^N \cosh(\omega z + b_k), \quad \beta, \omega \in \mathbb{R}, \quad \mathbb{C}, b_k \in \mathbb{C}.$$

The result is also valid for infinite sums

$$(3) \quad S(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \quad \lambda_n \in \mathbb{R}, \quad a_n \in \mathbb{C},$$

under conditions

$$(4) \quad \sum_{n=1}^{\infty} |a_n| < \infty, \quad \lambda_1 = \sup_n \lambda_n < \infty, \quad \lambda_2 = \inf_n \lambda_n > -\infty, \quad a_1 a_2 \neq 0.$$

Note that zeros of functions (1), (3) are located in a vertical strip of a finite width.

In 1949 M.G. Krein and B.Ja.Levin (see [5], also [6], ch.6, p.2 and Appendix 6) introduced and studied the class Δ of entire almost periodic functions of an exponential growth with zeros in a horizontal strip of a finite width. In particular, if a sum $S(z)$ in (3) satisfies condition (4), then $S(iz)$ belongs to Δ . In fact, representation (2) was obtained in [1] for functions from Δ with a discrete sets of differences between zeros (of course, it should be replaced $e^{\beta z}$ by $e^{i\beta z}$ and $\cosh(\omega z + b_k)$ by $\cos(\omega z + b_k)$).

The phenomenon takes place for a pair of functions and discreteness of the set of differences between their zeros as well ([3]). For example, let Q_1, Q_2 be entire functions of the form (3) under conditions (4). If the set $\{z - w, Q_1(z) = 0, Q_2(w) = 0\}$ is discrete, then the both functions have the form (2) with the same ω and possibly different C, β, N, b_k . In particular, in the case $Q_2(-z) = Q_1(z) = Q(z)$ we get representation (2) for any function (3) with a discrete set $\{z + z', Q(z) = Q(z') = 0\}$.

The proof of the above results are based on the property of zeros $\{z_j\}$ of functions $f \in \Delta$ ([6], Appendix 6, p.2) to be almost periodic in the sense of the following definition:

Definition 1. A zero set $\{z_j\}$ is almost periodic if for any $\varepsilon > 0$ there is a relatively dense set $E_\varepsilon \subset \mathbb{R}$ such that for each $\tau \in E_\varepsilon$ there exists a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ with the property

$$(5) \quad \sup_j |z_j + i\tau - z_{\sigma(j)}| < \varepsilon.$$

Recall that a set $E \subset \mathbb{R}$ is relatively dense, if there exists $L < \infty$ such that $E \cap [a, a + L] \neq \emptyset$ for any $a \in \mathbb{R}$.

Practically, the above definition is applied to zero sets in a *closed* vertical strip. In the present paper we investigate holomorphic almost periodic functions and almost periodic sets in an open strip, in particular, in the complex plane and in half-planes.

Definition 2. A continuous function $f(z)$ in a strip $S = \{z : a < \operatorname{Re} z < b\}$, $-\infty \leq a < b \leq \infty$ is almost periodic, if for any substrip $S_0, \overline{S_0} \subset S$ and any $\varepsilon > 0$ there is a relatively dense set $E_{\varepsilon, S_0} \subset \mathbb{R}$ such that for any $\tau \in E_{\varepsilon, S_0}$

$$(6) \quad \sup_{z \in S_0} |f(z + i\tau) - f(z)| < \varepsilon.$$

A typical example of an entire almost periodic function is sum (3) under conditions

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sup_n \lambda_n < \infty, \quad \inf_n \lambda_n > -\infty.$$

Moreover, a holomorphic function $f(z)$ in a strip $S = \{z : a < \operatorname{Re} z < b\}$, $-\infty \leq a < b \leq \infty$ is almost periodic if and only if there exists a sequence of quasipolynomials Q_n of the form (1) such that for any substrip $S_0, \overline{S_0} \subset S$,

$$\sup_{z \in S_0} |f(z) - Q_n(z)| \rightarrow 0, \quad n \rightarrow \infty.$$

(see [4], item 8).

In order to take into account multiplicities of zeros, we use the term *divisor* instead of zero set. Namely, a divisor Z in a domain D is a mapping $Z : D \rightarrow \mathbb{N} \cup \{0\}$ such that $|Z| = \operatorname{supp} Z$ is a set without limit points in D . In other words, a divisor in D is the sequence of points $\{z_j\} \subset D$ that has no limit points in D and every point of D appears at most a finite times in the sequence. If $Z(z) \leq 1$ for all $z \in D$, we will identify Z and $|Z|$. The divisor of zeros of a holomorphic function f in D is the map Z_f such that $Z_f(a)$ equals multiplicity of zero of function f at the point a .

The following definition appeared at first in [8]:

Definition 3 ([8],[2]). A divisor $Z = \{z_j\}$ in a strip $S = \{z : a < \operatorname{Re} z < b\}$, $-\infty \leq a < b \leq \infty$, is called almost periodic if for any $\varepsilon > 0$ and any substrip $S^0, \overline{S^0} \subset S$, there is a relatively dense set

$$(7) \quad E_{\varepsilon, S^0} = \{\tau \in \mathbb{R} : z_j \in S^0 \bigvee z_{\sigma(j)} \in S^0 \implies |z_j + i\tau - z_{\sigma(j)}| < \varepsilon\}.$$

where $\sigma = \sigma_\tau$ is a suitable bijection $\mathbb{N} \rightarrow \mathbb{N}$.

Remark. If we take $\sigma^{-1}(j)$ instead of j in (7), we get

$$(8) \quad z_j \in S^0 \bigvee z_{\sigma^{-1}(j)} \in S^0 \implies |z_j - i\tau - z_{\sigma^{-1}(j)}| < \varepsilon.$$

Theorem 1 ([2]). a) If f is a holomorphic function in a strip S with an almost periodic modulus, then the divisor of zeros of f is almost periodic,

b) for any almost periodic divisor Z in a strip S there is a holomorphic function f in S with an almost periodic modulus such that $Z = Z_f$,

c) for any almost periodic divisor Z in a strip S such that $|Z| \cap S' = \emptyset$ for some open substrip $S' \subset S$ there is a holomorphic almost periodic function f in S such that $Z = Z_f$.

There exist almost periodic divisors in the plane with a discrete set of differences that is not periodic.

Example 1. Let

$$Z = \{z_{n,k} = i2^{nk} + 2^k, n \in \mathbb{Z}, k \in \mathbb{N}\}$$

be a discrete set in \mathbb{C} . It is easy to see that the set is almost periodic, differences between its points form a discrete set, but Z is a countable union of periodic sets with different commensurable periods.

In our article we prove the following theorem

Theorem 2. *Let Z, W be almost periodic divisors in a strip $S = \{z : a < \operatorname{Re} z < b\}$, $-\infty \leq a < b \leq \infty$. If*

a) $|Z| \cap S' = \emptyset$ for some substrip $S' \subset S$,

b) for any substrip S_0 of a finite width, $\overline{S_0} \subset S$, the set $\{z - w, z \in |Z| \cap S_0, w \in |W| \cap S_0\}$ is discrete,

then Z, W are at most countable sums of periodic divisors with commensurable periods.

Show that condition a) is essential.

Example 2. Let $S = \{z : |\operatorname{Re} z| < 1\}$, $Z = \{z_{m,n} = (m + in)e^{i\alpha} \in S, m, n \in \mathbb{Z}\}$, where $\alpha \in \mathbb{R}$ such that $\cot \alpha$ is an irrational number. Clearly, the set of differences of elements of Z is discrete. Let us prove that Z is an almost periodic divisor without any periods.

By Kronecker Lemma (see, for example, [7], Ch.2, § 2), for any $\varepsilon > 0$ the inequalities

$$|t \cot \alpha| < \varepsilon \pmod{\mathbb{Z}}, \quad |t| < \varepsilon \pmod{\mathbb{Z}}$$

has a relatively dense set of common solutions. Therefore, the inequality

$$|m \cot \alpha| < \varepsilon(1 + |\cot \alpha|) \pmod{\mathbb{Z}}$$

has a relatively dense set of integer solutions. The later means that for any δ there exist pairs of integers $(m, n) \in \mathbb{Z}^2$ such that

$$(9) \quad |m \cos \alpha - n \sin \alpha| < \delta,$$

and the set of m with this property is relatively dense. Put

$$E = \{m \sin \alpha + n \cos \alpha : |m \cos \alpha - n \sin \alpha| < \delta\}.$$

Let $\tau = \tilde{m} \sin \alpha + \tilde{n} \cos \alpha \in E$. For any $z_{m,n} \in Z$ and $m' = m + \tilde{m}$, $n' = n + \tilde{n}$ we have

$$z_{m,n} + i\tau - z_{m',n'} = (m + in)e^{i\alpha} + i\operatorname{Im}[(\tilde{m} + i\tilde{n})e^{i\alpha}] - (m' + in')e^{i\alpha} = \operatorname{Re}[(\tilde{m} + i\tilde{n})e^{i\alpha}].$$

By (9), $|z_{m,n} + i\tau - z_{m',n'}| < \delta$. Since

$$|\tau - (\cos^2 \alpha / \sin \alpha + \sin \alpha)\tilde{m}| = |\tilde{n} \cos \alpha - (\cos^2 \alpha / \sin \alpha)\tilde{m}| < \delta |\cot \alpha|,$$

we see that the set E is relatively dense. So, Z is an almost periodic divisor.

Since Z sites in the vertical strip of width 2, we see that any period of Z must have the form iT , $T \in \mathbb{R}$. Hence, for some $(n, m), (n', m') \in \mathbb{Z}^2$, $(n, m) \neq (n', m')$,

$$(n + im)e^{i\alpha} - (n' + im')e^{i\alpha} = iT,$$

and

$$(n - n') + i(m - m') = iT(\cos \alpha - i \sin \alpha).$$

This equality contradicts our choice of α . Hence any part of Z has no periods.

Our proof of Theorem 2 makes into use the following Lemmas

Lemma 1. *Let $Z = \{z_n\}$ be an almost periodic divisor in a strip S , S^0 be a substrip, $\overline{S_0} \subset S$, $\varepsilon > 0$, $\tau_1, \tau_2 \in E_{\varepsilon, S^0}$, where E_{ε, S^0} is from Definition 3. Then there is a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that if $z_j \in S^0$, $\operatorname{dist}(z_j, \partial S^0) > \varepsilon$, then*

$$(10) \quad |z_j + i(\tau_1 - \tau_2) - z_{\sigma(j)}| < 2\varepsilon, \quad |z_j - i(\tau_1 - \tau_2) - z_{\sigma^{-1}(j)}| < 2\varepsilon.$$

It can be proved easily that the similar assertion is valid for $\tau_1 + \tau_2$ as well.

Proof. If z_j satisfies conditions of the Lemma, then for some bijection $\sigma_1 : \mathbb{N} \rightarrow \mathbb{N}$ we have $|z_j + i\tau_1 - z_{\sigma_1(j)}| < \varepsilon$, therefore $|\operatorname{Re}(z_{\sigma_1(j)} - z_j)| < \varepsilon$ and $z_{\sigma_1(j)} \in S^0$. By (8), we also get the inequality $|z_{\sigma_2^{-1} \circ \sigma_1(j)} - i\tau_2 - z_{\sigma_1(j)}| < \varepsilon$ for some bijection $\sigma_2 : \mathbb{N} \rightarrow \mathbb{N}$. Therefore, we obtain the first part in (10) with $\sigma = \sigma_2^{-1} \circ \sigma_1$. If we change places of τ_1 and τ_2 , we obtain the second part in (10).

Lemma 2. *Let $Z = \{z_n\}$, $W = \{w_m\}$ be almost periodic divisors in a strip S . Then for any $\varepsilon > 0$ and substrip $S^0 \subset S$, $\operatorname{dist}(S_0, \partial S) > \varepsilon$, there is a relatively dense set $E \subset \mathbb{R}$ with the property: for any $\tau \in E$ there are bijections σ_Z, σ_W from \mathbb{N} to \mathbb{N} such that for all $z_j \in |Z| \cap S^0$, $w_r \in |W| \cap S^0$*

$$(11) \quad |z_j + i\tau - z_{\sigma_Z(j)}| < \varepsilon, \quad |w_r + i\tau - w_{\sigma_W(r)}| < \varepsilon$$

and

$$(12) \quad |z_j - i\tau - z_{\sigma_Z^{-1}(j)}| < \varepsilon, \quad |w_r - i\tau - w_{\sigma_W^{-1}(r)}| < \varepsilon.$$

Proof. Put $\tilde{S} = \{z \in S : \operatorname{dist}(z, S^0) < \varepsilon/2\}$. By (7) and (8), there is a number L such that any interval of length L contains τ_Z and τ_W with the properties

$$(13) \quad |z_j + i\tau_Z - z_{\sigma_Z(j)}| < \varepsilon/4, \quad |z_j - i\tau_Z - z_{\sigma_Z^{-1}(j)}| < \varepsilon/4$$

for all $z_j \in |Z| \cap \tilde{S}$, and

$$(14) \quad |w_r + i\tau_W - w_{\sigma_W(r)}| < \varepsilon/4, \quad |w_r - i\tau_W - w_{\sigma_W^{-1}(r)}| < \varepsilon/4$$

for all $w_r \in |W| \cap \tilde{S}$. Here σ_Z, σ_W are some (in general, different) bijections $\mathbb{N} \rightarrow \mathbb{N}$.

We may suppose that $N = 2L/\varepsilon$ is an integer. Hence for any $k \in \mathbb{Z}$ there are integers $n(k), m(k)$, $0 \leq n(k), m(k) \leq N$ such that

$$|kL + n(k)\varepsilon/2 - \tau_Z| < \varepsilon/4, \quad |kL + m(k)\varepsilon/2 - \tau_W| < \varepsilon/4.$$

The differences $n(k) - m(k)$ take at most $2N + 1$ values. Choose k_1, \dots, k_r , $r \leq 2N + 1$, such that for any $k \in \mathbb{Z}$ there is k_s such that

$$n(k) - m(k) = n(k_s) - m(k_s).$$

It follows easily that any interval of length $L(\max_s(|k_s| + 2))$ contains a point of the form

$$\tau = Lk + n(k)\varepsilon/2 - Lk_s - n(k_s)\varepsilon/2 = Lk + m(k)\varepsilon/2 - Lk_s - m(k_s)\varepsilon/2.$$

If we replace τ_Z by $Lk + n(k)\varepsilon/2$ or $Lk_s + n(k_s)\varepsilon/2$ in (13), we obtain that this inequality satisfies with $\varepsilon/2$ instead of $\varepsilon/4$. The same is true if we replace τ_W by $Lk + m(k)\varepsilon/2$ or $Lk_s + m(k_s)\varepsilon/2$ in (14). Applying Lemma 1 with $S^0 = \tilde{S}$ and $\varepsilon/2$ instead of ε , we see that τ satisfies (11) and (12).

Proof of Theorem 2. Set a sequence of vertical substrips $S_k, \overline{S_k} \subset S_{k+1}$, such that

$$|Z| \cap S_1 \neq \emptyset, \quad |W| \cap S_1 \neq \emptyset, \quad S_1 \cap S' \neq \emptyset, \quad \cup_k S_k = S.$$

Take any points $z \in |Z| \cap S_1$, $w \in |W| \cap S_1$. It follows from (11) with $\varepsilon < \min\{\operatorname{dist}(z, \partial S_1), \operatorname{dist}(w, \partial S_1)\}$ that there is $R < \infty$ such that any horizontal strip of the width R contains at least one point $|Z| \cap S_1$ and at least one point $|W| \cap S_1$. By condition b) of the theorem, for each k the set $\{z - w, z \in |Z| \cap S_k, w \in |W| \cap S_k\}$ is discrete. Hence there exists $\gamma_k \in (0, 1/2)$ such that whenever

$$z_n, z_{n'} \in |Z| \cap S_{k+1}, \quad w_m, w_{m'} \in |W| \cap S_{k+1}, \quad z_n - w_m \neq z_{n'} - w_{m'},$$

$$|\operatorname{Im}(z_n - w_m)| < 2R + 3, \quad |\operatorname{Im}(z_{n'} - w_{m'})| < 2R + 3,$$

we get $\gamma_k < |(z_{n'} - w_{m'}) - (z_n - w_m)|$. In particular, if we put $w_m = w_{m'}$, then we get $\gamma_k < |z_n - z_{n'}|$ for any $z_n, z_{n'} \in |Z| \cap S_{k+1}$, $z_n \neq z_{n'}$.

Fix $z_n \in |Z| \cap S_k$, and let a number $\tau > 1$ satisfies (11) for Z and W with $\varepsilon = \gamma_k/2$. Then there is a unique $z_{n'} \in |Z|$ such that $|z_n + i\tau - z_{n'}| < \gamma_k/2$. Indeed, otherwise we obtain

$$|z_{n'} - z_{n''}| \leq |z_n + i\tau - z_{n'}| + |z_n + i\tau - z_{n''}| < \gamma_k.$$

Set $T_k = (z_{n'} - z_n)/i$.

Let $w_m \in |W|$ be such that $|\operatorname{Im}(w_m - z_n)| < 2R + 2$. By (11), there is a point $w_{m'} \in |W|$ such that $|w_m + i\tau - w_{m'}| < \gamma_k/2$. Therefore,

$$|(z_n - w_m) - (z_{n'} - w_{m'})| \leq |w_m + i\tau - w_{m'}| + |z_{n'} - z_n - i\tau| < \gamma_k.$$

Since

$$|\operatorname{Im}(z_{n'} - w_{m'})| \leq |\operatorname{Im}(z_n - w_m)| + |z_n - z_{n'} + i\tau| + |w_m - w_{m'} + i\tau| < 2R + 3,$$

we get $z_n - w_m = z_{n'} - w_{m'}$ due to the choice of γ_k . Therefore, $w_{m'} = w_m + iT_k$.

The latter equality takes place for all points $|W| \cap \{w : \operatorname{Im} z_n - 2R < \operatorname{Im} w < \operatorname{Im} z_n + 2R\}$, in particular, for some w_l such that $\operatorname{Im} z_n + R < \operatorname{Im} w_l < \operatorname{Im} z_n + 2R$. Namely, there is $w_{l'} \in |W|$ such that $w_{l'} = w_l + iT_k$. Let $\zeta \in |Z|$ be any point from the set

$$(15) \quad \{z : \operatorname{Im} z_n \leq \operatorname{Im} z < \operatorname{Im} z_n + 3R\} \subset \{z : \operatorname{Im} w_l - 2R < \operatorname{Im} z < \operatorname{Im} w_l + 2R\}.$$

By (11), there is a point $\zeta' \in |Z|$ such that $|\zeta + i\tau - \zeta'| < \gamma_k/2$. Therefore,

$$|(\zeta - w_l) - (\zeta' - w_{l'})| \leq |\zeta + i\tau - \zeta'| + |iT_k - i\tau| < \gamma_k.$$

Since $|\operatorname{Im} \zeta - \operatorname{Im} w_l| < 2R$ and

$$|\operatorname{Im}(\zeta' - w_{l'})| \leq |\operatorname{Im}(\zeta - w_l)| + |\zeta + i\tau - \zeta'| + |iT_k - i\tau| < 2R + 3,$$

we get $\zeta - w_l = \zeta' - w_{l'}$ due to the choice of γ_k . Therefore, $\zeta' = \zeta + iT_k$.

In particular, there is a point $z_s \in |Z| \cap \{z : \operatorname{Im} z_n + 2R \leq \operatorname{Im} z < \operatorname{Im} z_n + 3R\}$ such that $z_s + iT_k \in |Z|$. Continuing the line of reasoning, we obtain that for all $z \in |Z|$ such that $\operatorname{Im} z \geq \operatorname{Im} z_n$ we get $z + iT_k \in |Z|$ and for all $w \in |W|$ such that $\operatorname{Im} w \geq \operatorname{Im} z_n$ we get $w + iT_k \in |W|$.

If we take $w'_l \in |W|$ such that $\operatorname{Im} z_n - 2R < \operatorname{Im} w'_l < \operatorname{Im} z_n - R$, we also can find $w'_{l'} \in |W|$ such that $w'_{l'} = w'_l + iT_k$. Next, we prove that for any point $\tilde{\zeta} \in |Z| \cap \{z : \operatorname{Im} z_n - 3R \leq \operatorname{Im} z < \operatorname{Im} z_n\}$ there is $\tilde{\zeta}' \in |Z|$ such that $\tilde{\zeta}' = \tilde{\zeta} + iT_k$. Arguing as above, we show that for all $z \in |Z|$ such that $\operatorname{Im} z \leq \operatorname{Im} z_n$ we get $z + iT_k \in |Z|$ and for all $w \in |W|$ such that $\operatorname{Im} w \leq \operatorname{Im} z_n$ we get $w + iT_k \in |W|$.

Next, by (12), take for any $z \in |Z|$ a point $z'' \in |Z|$ such that $|z'' + i\tau - z| < \gamma_k/2$. Then $z'' + iT_k \in |Z|$ and

$$|(z'' + iT_k) - z| \leq |z'' + i\tau - z| + |i\tau - iT_k| < \gamma_k.$$

Therefore, $z'' + iT_k = z$ and $z - iT_k \in |Z|$ for all $z \in |Z|$. By the same arguments, $w - iT_k \in |W|$ for all $w \in |W|$.

If an imaginary part of T_k does not vanish, then either $z_n + iMT_k \in S'$, or $z_n - iMT_k \in S'$ for a suitable $M \in \mathbb{N}$. Therefore, T_k is real and for any $M \in \mathbb{Z}$ we get $(|Z| \cap S_k) + iMT_k = |Z| \cap S_k$, $(|W| \cap S_k) + iMT_k = |W| \cap S_k$. Hence, the restrictions $Z|_{S_k}$, $W|_{S_k}$ the divisors Z , W to S_k are periodic divisors with period iT_k .

The same arguments work for every $k = 1, 2, \dots$.

Let T_k^0 be the minimal common period of $Z|_{S_k}$ and $W|_{S_k}$. Clearly, $T_k/T_k^0 \in \mathbb{N}$. Besides, since $|Z| \cap S_k \subset |Z| \cap S_m$ for $m > k$, we have $T_m/T_k^0 \in \mathbb{N}$ as well.

Finally, let $Z_k = Z|_{S_{k+1} \setminus S_k}$, $W_k = W|_{S_{k+1} \setminus S_k}$. Then we obtain

$$Z = Z_1 + Z_2 + Z_3 + \dots, \quad W = W_1 + W_2 + W_3 + \dots$$

Theorem is proved.

Theorem 2 implies the corresponding result for almost periodic holomorphic functions.

Theorem 3. Let f, g be almost periodic functions in a strip $S = \{z : a < \operatorname{Re} z < b\}$, $-\infty \leq a < b \leq \infty$. If

- a) either f , or g has no zeros in an open substrip $S' \subset S$,
- b) for any substrip $S_0, \overline{S_0} \subset S$, the set $\{z - w, z, w \in S_0, f(z) = g(w) = 0\}$ is discrete, then

$$(16) \quad f(z) = f_0(z) \prod_{k=1}^{\infty} f_k(z), \quad g(z) = g_0(z) \prod_{k=1}^{\infty} g_k(z)$$

where f_0, g_0 are holomorphic almost periodic functions in S without zeros, f_k, g_k , $k = 1, 2, \dots$, are periodic holomorphic in S with commensurable periods T_k .

Proof. By Theorem 1a), the divisors Z, W of zeros of f, g , respectively, are almost periodic and satisfy other conditions of the previous theorem. Let S_k, Z_k, W_k , be the same as in the proof of Theorem 2. Set

$$S_k = \{z : \eta_k < \operatorname{Re} z < \eta'_k\}, \quad \eta_k < \eta_{k-1}, \quad \eta'_k > \eta'_{k-1}, \quad \forall k.$$

For any k there is only a finite number of points $a_1^k, \dots, a_{m_k}^k \in |Z_k| \cup \{z : 0 \leq \operatorname{Im} z < T_k\}$. Therefore, $|Z_k| = \{a_1^k, \dots, a_{m_k}^k\} + iT_k\mathbb{Z}$. Take $\varepsilon_k^j > 0$, $1 \leq j \leq m_k$, $3 \leq k < \infty$, such that

$$(17) \quad \sum_{j,k} \varepsilon_k^j < \infty.$$

Fix $k > 1$. Let

$$h_j(w) = 1 - w \exp(-2\pi a_j^k / T_k), \quad j = 1, \dots, m_k.$$

Clearly, $h_j(\exp(2\pi z / T_k))$ has a divisor $a_j^k + iT_k\mathbb{Z}$. Since $a_j^k \in S_k \setminus S_{k-1}$, we see that either $\eta'_{k-1} \leq \operatorname{Re} a_j^k$, or $\eta_{k-1} \geq \operatorname{Re} a_j^k$. In the first case, $\log h_j(w)$ is holomorphic on the disc $|w| < \exp(2\pi \operatorname{Re} a_j^k / T_k)$, and there is a polynomial P_j^k such that

$$|\log h_j(w) - P_j^k(w)| < \varepsilon_k^j \quad \text{for } |w| \leq \exp(2\pi \eta'_{k-2} / T_k).$$

In the second one, $\log h_j(w)$ is holomorphic on the set $|w| > \exp(2\pi \operatorname{Re} a_j^k / T_k)$, and there is a polynomial P_j^k such that

$$|\log h_j(w) - P_j^k(w^{-1})| < \varepsilon_k^j \quad \text{for } |w| \geq \exp(2\pi \eta_{k-2} / T_k).$$

Put $Q_j(z) = P_j^k(\exp(2\pi z / T_k))$ in the first case, and $Q_j(z) = P_j^k(\exp(-2\pi z / T_k))$ in the second one. We obtain

$$(18) \quad |h_j(e^{2\pi z / T_k}) e^{-Q_j(z)}| < e^{\varepsilon_k^j} \quad \text{for } z \in S_{k-2}.$$

Put

$$f_k(z) = \prod_{j=1}^{m_k} h_j(e^{2\pi z / T_k}), \quad k = 1, 2, \quad f_k(z) = \prod_{j=1}^{m_k} h_j(e^{2\pi z / T_k}) e^{-Q_j(z)}, \quad k > 2.$$

Clearly, $Z_{f_k} = Z_k$. By (17) and (18), the product in (16) converges in every substrip S_k . The function $f(z) / \prod_{k=1}^{\infty} f_k(z)$ is holomorphic in S , and, by [8], Theorem 1, it is almost periodic in S . In the same way, we obtain the representation of g .

Remark. It follows easily that for entire almost periodic functions f, g with zeros in a strip \tilde{S} of a finite width one can take $S_1 = \tilde{S}$. Therefore, f, g are periodic functions with the same period up to almost periodic multipliers without zeros. Hence the results of [1], [3] follow from Theorem 3.

REFERENCES

- [1] S.Ju.Favorov and N.P.Giryа, *On a criterion of periodicity of polynomials*, Ufa's Mathematical Journal 4, No.1 (2012), 47-52.
- [2] S.Yu. Favorov, A.Yu. Rashkovskii, A.I. Ronkin *Almost periodic divisors in a strip*, J. d'Analyse Math., Vol 74 (1998). P. 325–345.
- [3] N.P.Giryа *Periodicity of Dirichlet series*, Matematychni studii, v.42, 1, (2014), 13-18.
- [4] B. Jessen, H. Tornehave *Mean motions and zeros of almost periodic functions* Acta Math. 77. (1945), 137-279.
- [5] M.G.Krein and B.Ja.Levin, *On entire almost periodic functions of an exponential type*, DAN SSSR v.64 (1949), No 2, 285–287 (in Russian).
- [6] B.Ja.Levin, *Distributions of Zeros of Entire Functions*. Vol.5, Transl. of Math. Monograph, AMS Providence, R1, 1980.
- [7] B.M.Levitan *Almost periodic functions*, Gostehizdat, Moscow, 1953 (in Russian).
- [8] H. Tornehave, *Systems of zeros of holomorphic almost periodic functions*, Kobenhavns Universitet Matematisk Institut, Preprint no. 30, 1988.

SERGEY YURYEVICh FAVOROV,
 THE KARAZIN KHARKIV NATIONAL UNIVERSITY
 SVOBODA SQU., 4,
 61022, KHARKIV, UKRAINE
E-mail address: sfavorov@gmail.com